Holomorphic potential on symplectic toric manifolds

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1. Symplectic geometry

Definition. A pair (M, ω) is called symplectic manifold, if M is a smooth manifold and $\omega \in \Omega^2(M)$ is a closed non-degenerate 2-form.

It is immediate corollary from the definition that all symplectic manifolds are orientable and of even dimension. Here are some basic examples:

(1) $(\mathbb{R}^{2n}, \omega_0)$, where $\mathbb{R}^{2n}$ has coordinates $(x_1, y_1, \dots, x_n, y_n)$ and $\omega_0 = \sum_{k=1}^n dx_k \wedge dy_k$
(2) (T^*X, ω), where X – is any smooth manifold and $\omega = -d\alpha$, where α is a 1-form given by $\alpha_{(x,\xi)} = (d_{(x,\xi)}\pi)^*\xi$ at point $(x, \xi) \in T^*X$.

Definition. A map $f\colon (M, \omega_M) \to (N, \omega_N)$ of symplectic manifolds is called symplectomorphism if it is a diffeomorphism and $f^*\omega_N = \omega_M$.

Darboux theorem says that all symplectic manifolds of the same dimension are locally symplectomorhpic. In other words, symplectic manifolds do not have local invariants (such as curvature in Riemannian geometry). Thus to understand something about these objects we need to study global invariants. Lagrangian submanifolds may play such role.

Definition. Let (M^{2n}, ω) be a $2n$-dimensional symplectic manifold. A submanifold X of M is called Lagrangian if, at each point $p \in X$, $\omega_p|_{T_pX} \equiv 0$ and $\dim T_pX = n$.

2. Almost complex structure

Definition. A vector space V is said to have almost complex structure if there is an endomorphism $J\colon V \to V$ such that $J^2 = -\mathrm{Id}$

Definition. A manifold M is said to be endowed with almost complex structure if there is a smooth family $\{J_p\}_{p\in M}$ of almost complex structures for each T_pM. The pair (M, J) is called almost complex manifold.

Definition. Let j be a complex structure on Riemannian surface Σ^2.
Pseudo-holomorphic curve is a map $u\colon (\Sigma^2, j) \to (M, J)$ such that

$$du \circ j = J \circ du$$

3. Compatible structures. Kähler potential

Definition. Let (M, ω) be a symplectic manifold. An almost complex structure J on M is called compatible with ω (or ω-compatible) if $g(u, v) := \omega(u, Ju)$ is a Riemannian metric. The triple (ω, g, J) is called compatible triple.

Definition. A Kähler manifold is a symplectic manifol (M, ω) equipped with an integrable compatible almost complex structure. The symplectic form ω is then called a Kähler form.

4. Symplectic toric varieties

4.1. **Moment map.**

Definition. Let (M, ω) be a symplectic manifold. We say that there is a symplectic action of Lie group G on (M, ω) if there exists a homomorhphism

$$\psi \colon G \to \mathrm{Symp}(M, \omega) \subset \mathrm{Diff}(M)$$

Definition. Let $\mathfrak{g}$ be a Lie algebra of Lie group G. The action ψ is a hamiltonian action on symplectic manifold (M^{2n}, ω) if there exists a map

$$\mu \colon M \to \mathfrak{g}^*$$

such that

(1) For each $X \in \mathfrak{g}$ let
- $\mu^X \colon M \to \mathbb{R}$, $\mu^X(p) := \langle \mu(p), X \rangle$
- $X^{\#}$ be a vector field on M generated by $\{\exp tX | t \in \mathbb{R}\} \subset G$

Then

$$d\mu^X = \iota_{X^{\#}}\omega$$

(2) For all $g \in G$

$$\mu \circ \psi_g = \mathrm{Ad}_g^* \circ \mu$$

The map μ is called moment map for a hamiltonian G-space (M, ω, G, μ).

We will be interested only in the case when $G = \mathbb{T}^n$, i.e. our Lie group is a torus of half dimension of symplectic manifold.

The Atiyah-Guillemin-Sternberg theorem tells that for $G = \mathbb{T}^m$ the image $\mu(M)$ of the moment map is a convex polytope which is called moment polytope.

Definition. A $2n$-dimensional symplectic toric manifold is a compact connected symplectic manifold (M^{2n}, ω) equipped with an effective hamiltonian action of an n-torus $\mathbb{T}^n$ and with a corresponding moment map $\mu : M \to \mathbb{R}^n$.

It turns out that for the symplectic toric manifolds the moment polytope is of very special type.

4.2. **Delzant polytopes.**

Definition. A Delzant polytope $P \subset \mathbb{R}^n$ is a convex polytope satisfying:
- it is simple, i.e. there are n edges meeting at each vertex;
- it is rational, i.e., the edges meeting at the vertex p are rational in the sense that each edge is of the form $p + tu_i$, $t \geq 0$, where $u_i \in \mathbb{Z}^n$;
- t is smooth, i.e., for each vertex, the corresponding $u_1, \ldots, u_n$ can be chosen to be a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

There is remarkable theorem of Delzant.

Theorem 1 (Delzant)**.** *Symplectic toric manifolds are classified by Delzant polytopes. More specifically, there is the following one-to-one correspondence*

$$\{\textit{Symplectic toric manifolds}\} \longleftrightarrow \{\textit{Delzant polytopes}\}$$
$$(M^{2n}, \omega, \mathbb{T}^n, \mu) \longmapsto \mu(M)$$

5. Kähler potential for toric varieties

A Delzant polytope P can be described by a set of inequalities of the form $\langle x, v_r\rangle \geq \lambda_r, r = 1, \dots, d$, where d is the number of faces of Delzant polytope P, each v_r being a primitive element of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ and inward-pointing normal to the r-th $(n-1)$-dimensional face of P. Consider the affine functions $\ell_r \colon \mathbb{R}^n \to \mathbb{R}, r = 1, ..., d$, defined by

$$\ell_r(x) = \langle x, v_r\rangle - \lambda_r$$

Then $x \in \mathring{P}$ if and only if $\ell_r(x) > 0$ for all r and hence the function

$$g_P(x) = \frac{1}{2}\sum_{j=1}^{d} \ell_r(x)\log(\ell_r(x)) \quad (\heartsuit)$$

is smooth on $\mathring{P}$.

Theorem 2 (Guillemin)**.** *The "canonical" compatible complex structure on toric symplectic manifold (M^{2n}, ω) is given in symplectic coordinates (x, y) of $\mathring{M} \cong \mathring{P} \times \mathbb{T}^n$ by*

$$J_P = \begin{pmatrix} 0 & -Hess(g_P)^{-1} \\ Hess(g_P) & 0 \end{pmatrix}$$

6. Problem statement

Theorem 3. *Let (M_P, ω_P, μ_P) be the toric symplectic manifold associated to a Delzant polytope $P \subset \mathbb{R}^n$, and J any compatible toric complex structure. Then J is determined by a potential $g \in C^\infty(\mathring{P})$ of the form*

$$g = g_P + h$$

where g_P is given by $(\heartsuit)$, h is smooth on the whole P, and the matrix $G = Hess_x(g)$ is positive definite on $\mathring{P}$.

6.1. **Further plan.** Since for a given toric symplectic manifold we know everything about its compatible complex structures, we can study subobjects of these to structures: Lagrangian submanifolds and pseudo-holomorphic curves.

The idea is the following. In Delzant polytope we consider some set (namely a tropical curve) and then we want to lift in our manifold in two ways: one way to a Lagrangian submanifold and the other to a pseudo-holomorhic curve.