OUTER AUTOMORPHISMS OF UNIFORM LATTICES IN COMPLEX LIE GROUPS

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It turns out that finite subgroups of various complicated groups coming from geometry (such as automorphism groups of projective varieties, Cremona groups, mapping class groups, etc) satisfy certain "boundedness" properties. More precisely, one can consider the following ones.

Definition 1 (V. L. Popov)**.** Let G be a group. We say that G is *Jordan* if there exists $J(G) \in \mathbb{N}$ such that for any finite subgroup $H \subset G$ there exists a normal abelian subgroup $A \lhd H$ of index at most $J(G)$. We say that G has *bounded finite subgroups* if there exists $B(G)$ such that for any finite subgroup $H \subset G$ one has $|H| \leqslant B(G)$.

The most important examples of Jordan groups are $\mathrm{GL}_n(\mathbb{C})$ (Jordan's theorem) and, more generally, connected Lie groups (Boothby–Wang; Popov). As for infinite groups with bounded finite subgroups, the basic example is $\mathrm{GL}_n(\mathbb{Z})$ (Minkowski's theorem).

Let X be a compact complex manifold. We denote by $\mathrm{Aut}(X)$ the group of biholomorphic automorphisms of X. Then there is an exact sequence

$$1 \to \mathrm{Aut}^0(X) \to \mathrm{Aut}(X) \to \mathrm{Aut}^*(X) \to 1,$$

where the connected component of the identity $\mathrm{Aut}^0(X)$ is a complex Lie group (Bochner–Montgomery) and $\mathrm{Aut}^*(X)$ is a discrete group, called the group of connected components of $\mathrm{Aut}(X)$.

For a general compact complex manifold X little is known about the group $\mathrm{Aut}^*(X)$. However, for X projective (or compact Kaehler) the group $\mathrm{Aut}^*(X)$ is an extension of a linear (over $\mathbb{Z}$) group by a finite group. This allows to prove the following result.

Theorem 2 (S. Meng–D.-Q. Zhang; J. H. Kim)**.** *Let X be a compact Kaehler manifold. Then the group $\mathrm{Aut}(X)$ is Jordan.*

I would like to prove a similar result for automorphism groups of (non-Kaehler) compact complex manifolds from a particular class [Wa54].

Definition 3. A compact complex parallelizable manifold X is a quotient G/Γ of a connected complex Lie group G by a uniform lattice $\Gamma \subset G$.

These manifolds were extensively studied by J. Winkelmann [Win98]. In particular, their automorphism groups can be explicitely described in terms of G and Γ.

Proposition 4 (J. Winkelmann)**.** *Let G be a simply-connected Lie group and let $\Gamma \subset G$ be a uniform lattice. Then the group $\mathrm{Aut}^*(G/\Gamma)$ embeds into the group of outer automorphisms $\mathrm{Out}(\Gamma)$.*

The main result of [Gol23] is the following theorem.

Theorem 5 (G., 2023)**.** *Let Γ be a uniform lattice in a connected complex Lie group G. Then the orders of finite subgroups in the group $\mathrm{Out}(\Gamma)$ of outer automorphisms of Γ are bounded.*

The main idea of the proof is to use the following exact sequence:

$$1 \to \Gamma \cap R \to \Gamma \to \Gamma/\Gamma \cap R \to 1,$$

where R is the radical of G. Then $\Gamma \cap R$ is a uniform lattice in R and the quotient group $\Gamma/\Gamma \cap R$ is a uniform lattice in the semisimple group $S = G/R$. We show that the result holds for lattices in solvable Lie groups (outer automorphisms of polycyclic groups) and semisimple Lie groups (Mostow's strong rigidity and Weil's local rigidity). Then we use group cohomology (automorphisms of group extensions) to treat the general case. We refer to the excellent survey [VGS88] for the aforementioned results on lattices in Lie groups.

As a corollary, we obtain the following result ([Gol23, Theorem 1.6]).

Corollary 6. *Let X be a compact complex parallelizable manifold. Then the group* $\mathrm{Aut}(X)$ *is Jordan.*

An interesting question for further research is to find an effective Jordan constant in terms of certain invariants of the Lie group and the lattice Γ.

Question 7. Let S be a complex semisimple Lie group and $\Gamma \subset S$ a uniform lattice. Let M be a $\mathbb{Z}\Gamma$-module, finitely generated as an abelian group. Is it possible to bound the order of

$$H^1(\Gamma, M)_{tors}$$

by a function of $\dim(S)$, $\mathrm{vol}(S/\Gamma)$ and the number of generators of M?

References

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