Severi-Brauer varieties and central simple algebras

1 Central simple algebra

A ***central simple algebra*** is a ***central*** ***simple*** ***algebra***, i.e. the notion is separable in a sense that it's a combination of several definitions.
When one speaks about an algebra in the context of our topic he assumes that everything is being done over commutative ring with 1, usually denoted as R. Either leftness or rightness of module is chosen as well. So, here are equivalent definitions of an algebra:

- An R-algebra is an R-module, equipped with a proper bilinear map (so that we can call it a *multiplication*) [1, p. 1]
- An R-algebra is a ring together with a homomorphism from R to its center. [1, p.2]

An algebra is ***simple*** equals any of:

- the only ideals are the trivial one and the whole ring [1, p. 44]
- it is some matrix algebra over some division ring [1, p. 49-50]

And ***central*** iff center equals to the ring. [1, p. 224]
It turns out that the center of a simple algebra is a field, so we may consider algebras over fields without loss of generality. [1, p. 219]

2 Brauer group

Brauer group is a **Brauer** **group**. That is, we have something that is possible to turn into a group in essential way. One can consider the following approaches:

- Essential way: F — field, $\mathfrak{S}(F)$ — all central simple algebras over F. Via tensor product we split $\mathfrak{S}(F)$ into equivalence classes: for $A, B \in \mathfrak{S}(F)$: $A \sim B \iff A, B$ are matrix algebras over the same division ring. The arose equivalence classes are the elements of Brauer group with tensor product as an operation. Denotion: $Br(F)$ [1, p. 227]
- Brauer group is the second cohomology group: $Br(F) = H^2(Gal(F), (F^{sep})^*)$ [2, p. 67]
- Slightly different notation: $Br(F) = \varinjlim H^2(G(E/F), E^\circ)$ [1, p. 267]

Can be rewritten via Ext functor as well. [2, p. 16] (Ext ⇜⇝ cohomologies; Tor ⇜⇝ homologies)

3 Severi-Brauer variety

Several ways to define are possible as well:

- Severi-Brauer variety is a form of projective space [2, p. 46]
- Severi-Brauer variety (over field F) is a scheme X st there exists a separable field extension E/F st $X \times_{\text{Spec}F} \text{Spec}E$ is a projective space over E [3, p. 23]

4 Amitsur conjecture

This statement determines a connection between Severi-Brauer varieties and Brauer groups:

- two Severi-Brauer varieties are birationally isomorphic iff the two corresponding algebras have the same degree and generate the same cyclic subgroup in the Brauer group ($\Longrightarrow$ is proved in general and $\Longleftarrow$ is proved only for certain cases, so the conjecture says that $\Longleftarrow$ holds in general) [2, p. 80], [3, p. 30]

My research concerns $\Longleftarrow$ case.

References

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