DISCRIMINANTS OF GENERAL POLYNOMIAL SYSTEMS

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1 Geometry of finite sets on a lattice

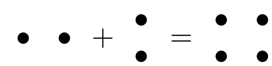
$\mathscr{A} = (A_1, ..., A_m)$, A_i is a finite set from $\mathbb{Z}^n$ and $0 \in A_i$;

$\mathfrak{c}(\mathscr{A}) = \#(\text{sets in the tuple } \mathscr{A})$;

$\langle \mathscr{A} \rangle = min\,\{L \text{ - affine sublattice of } \mathbb{Z}^n \mid A_i \subset L\}$.

Definition 1. The *mixed volume* MV of a tuple $\mathscr{A}$ of finite sets in the lattice L is defined as

$$MV_L(\mathscr{A}) = \frac{(-1)^{\mathfrak{c}(\mathscr{A})}}{\mathfrak{c}(\mathscr{A})!} \sum_{\mathscr{B} \subseteq \mathscr{A}} (-1)^{\mathfrak{c}(\mathscr{B})} Vol_L\,(\mathscr{B})\,, \tag{1}$$

where $Vol_L\,(\mathscr{B}) = Vol_L(\sum\limits_{B \in \mathscr{B}} B)$. (Example of Minkowski sum: )

A tuple $\mathscr{A}$ of finite sets is *irreducible* if

$$\forall \mathscr{B} \subset \mathscr{A}\ codim_{\langle \mathscr{A} \rangle} \langle \mathscr{B} \rangle + \mathfrak{c}(\mathscr{B}) < \mathfrak{c}(\mathscr{A}), \tag{2}$$

A tuple $\mathscr{A}$ is called BK if $dim\,\langle \mathscr{A} \rangle = \mathfrak{c}(\mathscr{A})$.

Theorem 1. *Consider a reducible BK-tuple $\mathscr{A}$ in the lattice L with positive mixed volume $MV_L(\mathscr{A}) > 0$. Then the following holds:*

1) irreducible subtuples of $\mathscr{A}$ don't intersect;

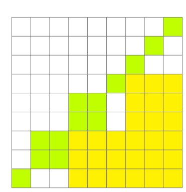
2) $\pi(\mathscr{A} \backslash \mathscr{B})$ is a BK-tuple with positive mixed volume, where $\mathscr{B}$ is an BK-subtuple and $\pi : L \to L/\overline{\langle \mathscr{B} \rangle}$.

$$MV_L(\mathscr{A}) = s(\mathscr{M}) \cdot MV_{L/\overline{\langle \mathscr{M} \rangle}} \pi(\mathscr{A} \backslash \mathscr{M}) \cdot \prod_{\mathscr{B} \in ir(\mathscr{A})} MV_{\overline{\langle \mathscr{B} \rangle}}(\mathscr{B}), \tag{3}$$

$$ir(\mathscr{A}) = \{\mathscr{B} \subseteq \mathscr{A} \mid \mathscr{B} \text{ is an irreducible BK-subtuple}\}, \tag{4}$$

$$\mathscr{M} = \bigsqcup_{\mathscr{B} \in ir(\mathscr{A})} \mathscr{B}. \tag{5}$$

$$\begin{array}{ccccccccc} \mathscr{A}_0 & \xrightarrow{q_1} & \mathscr{A}_1 & \xrightarrow{q_2} & ... & \xrightarrow{q_{d-1}} & \mathscr{A}_{d-1} & \xrightarrow{q_d} & \mathscr{A}_d \\ \cup & & \cup & & & & \cup & & \cup \\ ir(\mathscr{A}_0) & & ir(\mathscr{A}_1) & & ... & & ir(\mathscr{A}_{d-1}) & & ir(\mathscr{A}_d) \end{array}$$



$$\mathscr{A} = \bigsqcup_{i=0}^{d} \left(\bigsqcup_{\mathscr{C} \subset \mathscr{A} : \pi_i(\mathscr{C}) \in ir(\mathscr{A}_i)} \mathscr{C} \right)$$

$$MV_L(\mathscr{A}) = \prod_{i=0}^{d} \left(s(\mathscr{M}_i) \prod_{\mathscr{B} \subset ir(\mathscr{A}_i)} MV_{\langle \mathscr{B} \rangle} \mathscr{B} \right)$$

2 Discriminants

$(\mathbb{C}\backslash 0)^A = \{\sum_{a\in A} c_a x^a \mid A \subset \mathbb{Z}^n,\ c_a \in \mathbb{C}\backslash 0\}$; $(\mathbb{C}\backslash 0)^{\mathscr{A}} = (\mathbb{C}\backslash 0)^{A_1} \oplus ... \oplus (\mathbb{C}\backslash 0)^{A_m}$.

Definition 2. Let $\Phi = (\varphi_1, ..., \varphi_m) \in (\mathbb{C}\backslash 0)^{\mathscr{A}},\ y \in (\mathbb{C}\backslash 0)^n$. A point y is a *singular point* of a system Φ if y is a common root of the equations $\{\varphi_j(y) = 0\}_j$, and $\{d\varphi_j(y)\}_{j\in\overline{1,m}}$ are linearly dependent.

$\mathscr{A}$*-discriminant* is the algebraic set $D_{\mathscr{A}}$ in the space of polynomial systems $(\mathbb{C}\backslash 0)^{\mathscr{A}}$,

$$D_{\mathscr{A}} = \overline{\{\Phi \in (\mathbb{C}\backslash 0)^{\mathscr{A}} \mid \Phi \text{ has a singular point}\}}. \tag{6}$$

Proposition 1. *Any element g of $\mathsf{AGL}(n, \mathbb{Z})$ define an isomorphism of the discriminants for the given tuple $\mathscr{A}$:*

$$D_{\mathscr{A}} \cong D_{g\mathscr{A}}. \tag{7}$$

$$\Phi \begin{cases} P(x,y) = \sum_{j=0}^{n} a_j(x) y^j = 0, \quad n > 0, \\ P_m(x) = 0. \end{cases} \tag{8}$$

Lemma 1. *For systems of type 8, the discriminant $D_{\mathscr{A}}$ is not a hypersurface in all cases except for:*

1) $|supp\, P_m| = 2$, and $n > 3$;

2) $|supp\, P_m| > 2$, and $n = 2$, $|supp\, a_0| = 1$, or $|supp\, a_1| = 1$.

Theorem 2. *Dual defective tuples of two sets in $\mathbb{Z}^2$ are the pair of standard triangles and those, described by Lemma 1.*

3 Discriminants of BK-systems

$$\Lambda(\mathscr{A}) = \{\mathscr{B} \in ir\, \mathscr{A} \mid \mathscr{B} \text{ is a linear BK-subtuple}\}, \tag{9}$$

$$\Lambda_1(\mathscr{A}) = \{\mathscr{B} \in \Lambda(\mathscr{A}) \mid \mathfrak{c}(\mathscr{B}) = 1\}. \tag{10}$$

Proposition 2. *The discriminant $D_{\mathscr{A}}$ for linear systems with reducible support $\mathscr{A}$ that has positive mixed volume is a collection of $\sum_{i=0}^{d} (|\Lambda(\mathscr{A}_i)| - |\Lambda_1(\mathscr{A}_i)|)$ components of codimension 2. In other words,*

$$D_{\mathscr{A}} = \bigcup_{i=0}^{d} \boldsymbol{\pi}_i^{-1}(D_{\mathscr{M}_i}), \tag{11}$$

where $\boldsymbol{\pi}_i$ is a restriction map from $(\mathbb{C}\backslash 0)^{\mathscr{A}}$ to $(\mathbb{C}\backslash 0)^{\mathscr{M}_i}$.

Proposition 3. *For systems with semi-irreducible support $\mathscr{A}$, the discriminant $D_{\mathscr{A}}$ is a collection of $|ir\, \mathscr{A}| - |\Lambda(\mathscr{A})|$ hypersurfaces and $|\Lambda(\mathscr{A})| - |\Lambda_1(\mathscr{A})|$ components of codimension 2,*

$$D_{\mathscr{A}} = \bigcup_{\mathscr{B} \in ir\, \mathscr{A}} \left(D_{\mathscr{B}} \times (\mathbb{C}\backslash 0)^{\mathscr{A}\backslash\mathscr{B}}\right). \tag{12}$$