Convergence Domain of Hpergeometric Series for Solutions to The System of Algebraic Equations

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History of the topic

In 1889 J. Horn gave a description of convergence domain for the hypergeometric series in two variables a, b. In fact he introduced a parameterization

$$|a| = \varphi_1(s), |b| = \varphi_2(s), s \in \mathbb{R}_+$$

for the conjugate radii of convergence domain.

After Horn, at the beginning of the $20-th$ century, Hj. Mellin and R. Birkeland introduced the hypergeometric functions presenting the solutions to a general algebraic equation. In 1990 I. M. Gelfand, M.M. Kapranov and A.V. Zelevinsky turned back to the topic and proposed the relation between the hypergeomteric functions and A-discriminants, a generalization for classical discriminant. In 1991 M. Kapranov gave the concept 'Horn uniformization', an analog for the Horn's function $\mathcal{H}$, a parameterization for the envelop of the convergence domain for a hypergeometric series.

Resent researches

Deeper in the situation, in 2004 M. Passare and A. K. Tsikh asserted the concept uniformization of the discriminant locus or Horn–Kapranov parametrization, a generalization for Horn's parameterization. They claimed the Horn–Kapranov parametrization parameterizes the discriminant hypersurface of the equation

$$a_0 + a_1 y + \cdots + y^p + \cdots + y^q + \cdots + a_{n-1} y^{n-1} + a_n y^n = 0, \quad (1)$$

where p and q are two fixed different nonnegative integers, and described it as a $(q-p)$-fold covering of projective space $\mathbb{CP}^{n-2}$. After that, A. K. Tsikh and I. A. Antipova developed this parameterization for the system of algebraic equation case in 2012.

Our research

The image of the space $\mathbb{CP}^{n-2}$ under the Horn-Kapranov parameterization is the discriminant hypersurface associate with the equation (1). This observation facilitates the determination of the convergence domain for hypergeometric functions. This image interests us to get more insight. We study the convergence domains of hypergeometric series presenting solutions to a system of two algebraic trinomial equations with two unknown, i.e. the solution $y = (y_1, y_2)$ to the system

$$\begin{cases} a_1 y^{\alpha_1} + a_2 y^{\alpha_2} + a_3 y^{\alpha_3} = 0, \\ b_1 y^{\beta_1} + b_2 y^{\beta_2} + b_3 y^{\beta_3} = 0, \end{cases} \quad (2)$$

with variable coefficients $a_j, b_j \in \mathbb{C}^2$ and fixed exponents $\alpha_j, \beta_j \in \mathbb{Z}^2$.

Using suitable division we can reduce each such trinomial equation and then obtain a reduced system of the form

$$\begin{cases} y_1^m + a y_1^p y_2^q - 1 = 0, \\ y_2^l + b y_1^u y_2^v - 1 = 0, \end{cases} \quad (3)$$

where $m, n \in \mathbb{Z}_{>0}$ and $(p, q), (u, v) \in \mathbb{Z}^2$.

The discriminant for the general system (2) is a polynomial denoted by $\Delta(a_1, a_2, a_3, b_1, b_2, b_3)$. For the reduced system (3) the discriminant is the polynomial $\Delta(1, a, -1, 1, b, -1)$. For the solution $y = (y_1, y_2)$ to (3) with $(\mu_1, \mu_2), (p, q), (u, v) \in \mathbb{Z}^2_{>0}$, we consider the hypergeometric series for the monomial function

$$y_1^{\mu_1} y_2^{\mu_2} = \sum_{\alpha \in \mathbb{N}^2} c_\alpha a^{\alpha_1} b^{\alpha_2}. \quad (4)$$

And for the so-called principal solution (satisfying initial condition $y(0,0) = (1,1)$), we get the following expression of coefficients c_α:

$$c_\alpha = (-1)^{\alpha_1 + \alpha_2} \cdot \Gamma_\alpha \cdot \mathrm{R}_\alpha,$$

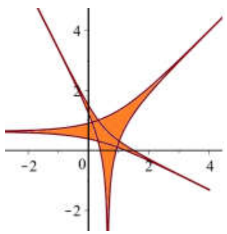
where

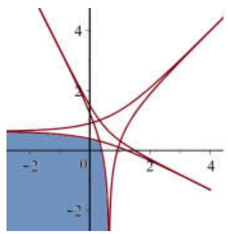
$$\Gamma_\alpha = \frac{\Gamma(\frac{\mu_1+m}{m} + \frac{p}{m}\alpha_1 + \frac{u}{m}\alpha_2)\Gamma(\frac{\mu_2+l}{l} + \frac{q}{l}\alpha_1 + \frac{v}{l}\alpha_2)}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\Gamma(\frac{\mu_1+m}{m} + \frac{p-m}{m}\alpha_1 + \frac{u}{m}\alpha_2)\Gamma(\frac{\mu_2+l}{l} + \frac{q}{l}\alpha_1 + \frac{v-l}{l}\alpha_2)},$$

$$\mathrm{R}_\alpha = \frac{(\mu_1 + u\alpha_2)(\mu_2 + q\alpha_1) - uq\alpha_1\alpha_2}{(\mu_1 + p\alpha_1 + u\alpha_2)(\mu_2 + q\alpha_1 + v\alpha_2)}.$$

We call Γ_α the gamma-part and R_α the rational-part of the coefficient c_α.

Amoebas





Theorem

Let $\Delta(1, a, -1, 1, b, -1)$ be the discriminant of the system (3). The convergence domain of the hypergeometric series (4) representing the monomial $y_1^{\mu_1} y_2^{\mu_2}$ of the principal solution is determined by 1 or 2 or 3 inequalities of the types

$$\pm\Delta(\pm 1, \pm|a|, -1, \pm 1, \pm|b|, -1) < 0. \quad (5)$$

Example

Consider the system of trinomial equations

$$\begin{cases} y_1^3 + a y_1 y_2 - 1 = 0, \\ y_2^3 + b y_1 y_2 - 1 = 0. \end{cases} \quad (6)$$

Completing it by the Jacobian equation $J(y_1, y_2) = 0$ and eliminating the variables y_1 and y_2 we obtain the following expression for the discriminant $\Delta(1, a, -1, 1, b, -1) =: \Delta(a, b)$:

$$\Delta(a, b) = -27 - 4a^3 + 6a^2 b + 6ab^2 - 4b^3 + a^4 b^2 - 2a^3 b^3 + a^2 b^4. \quad (7)$$

The power series for the monomial $y_1^{\mu_1} y_2^{\mu_2}$ of the principal solution to the system (6) is defined by gamma and rational parts

$$\Gamma_\alpha = \frac{\Gamma(\frac{\mu_1+3}{3} + \frac{\alpha_1}{3} + \frac{\alpha_2}{3})\Gamma(\frac{\mu_2+3}{3} + \frac{\alpha_1}{3} + \frac{\alpha_2}{3})}{\Gamma(\frac{\mu_1+3}{3} - \frac{2\alpha_1}{3} + \frac{\alpha_2}{3})\Gamma(\frac{\mu_2+3}{3} + \frac{\alpha_1}{3} - \frac{2\alpha_2}{3})},$$

$$R_\alpha = \frac{(\mu_1\mu_2 + \mu_1\alpha_1 + \mu_2\alpha_2)}{(\mu_1 + \alpha_1 + \alpha_2)(\mu_2 + \alpha_1 + \alpha_2)}.$$

The convergence domain of the considered series is determined by two inequalities of the type (5) as following:

$$D = \{\Delta(|a|, -|b|) < 0\} \cap \{\Delta(-|a|, |b|) < 0\}.$$

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