Suzdal'

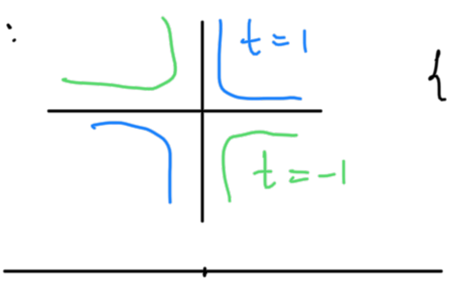
Infinitesimal deformations of schemes

– R. Hartshorne „Deformation theory"

<u>definition</u> a family of varieties $\{X_t\}_{t\in T}$ is a flat morphism of schemes $/k$ ← alg. closed field

$$\begin{array}{ccc} X_t = \mathfrak{X}_t & \subset & \mathfrak{X} \\ \downarrow & & \downarrow \text{flat} \\ t & \in & T \end{array}$$

<u>example</u> :



$$\begin{array}{c} \{xy=0\} = X_0 \subset \\ \{xy=t\} = X_t \subset \end{array} \mathbb{A}^2 \sim k[x,y]$$
$$\downarrow$$
$$\mathbb{A}^1 \sim k[x]$$

Want to study infinitesimal behaviour of families near a fixed fiber X_0

<u>definition</u>
Let X - scheme over k, $Y \subseteq X$ closed subscheme

A deformation (infinitesimal) of Y over Spec D, $D = k[t]/t^2$ in X is a closed subscheme $Y' \subset X' = X \times \operatorname{Spec} D$ which is flat over D.

<u>Theorem</u> Let X - scheme over k, $Y \subseteq X$ closed subscheme

Then

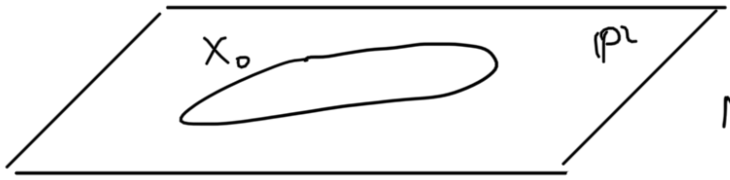
$$\left\{ \begin{array}{c} \text{first order deformations of } Y \\ \text{over } D \text{ in } X \end{array} \right\} \longleftrightarrow H^0(Y, N_{Y/X})$$

$\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}_Y/\mathcal{I}_Y^2, \mathcal{O}_Y)$ normal sheaf

<u>Corollary</u> If X_0 closed subscheme of $\mathbb{P}^n_k = X$

then
$$\left\{ \begin{array}{c} \text{Zariski tangent space} \\ \text{at } \underset{\sim X_0}{h_0} \in H \end{array} \right\} \simeq H^0(X_0, N_{X_0/X})$$

<u>Example</u>: smooth plane conics



$H = \mathbb{P}^5$

$$N_{X_0/X} = \mathcal{H}om_{\mathbb{P}^2}(\mathcal{I}, \mathcal{O}_{X_0})$$

$\mathcal{I} = \mathcal{O}_{\mathbb{P}^2}(-2)$, $\mathcal{I}/\mathcal{I}^2 = \mathcal{O}_{X_0}(-2)$, $X_0 \simeq \mathbb{P}^1$

$$= \mathcal{H}om(\mathcal{O}_{X_0}(-2), \mathcal{O}_{X_0})$$
$$= \mathcal{O}_{\mathbb{P}^1}(4)$$

$$\Longrightarrow \dim_k \mathbb{P}^5_k = \dim H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4)) = 5$$

More generally, let A - local Artin k-algebra.
$\ni m$ $\quad A/m \simeq k$

<u>definition</u>: Let X_0 - scheme $/k$. A <u>deformation of X_0 over A</u> is a scheme X flat over A, together with a closed immersion $i : X_0 \hookrightarrow X$

s.t. $X_0 \xrightarrow[L\times_A k]{\sim} k \times_A X'$ - iso

$$\begin{array}{ccc} X_0 & \hookrightarrow & X \\ \downarrow & & \downarrow \\ k & \hookrightarrow & A \end{array}$$

Two deformations X_1', X_2' are equivalent if

$\exists\, f$
$$\begin{tikzcd} X_1' \arrow[rr, "\sim"] \arrow[dr] & \arrow[d, phantom, "G"] & X_2' \arrow[dl] \\ & A & \end{tikzcd} \quad + \quad \begin{tikzcd} X_1' \arrow[r, "f"] \arrow[dr, phantom, "G"] & X_1' \\ & X_0 \arrow[ul, "\iota_1"] \arrow[u, "\iota_2"'] \end{tikzcd}$$

Affine case: Let B_0 - k-algebra, $X_0 = \operatorname{Spec} B_0$

Deformation of X_0 over D is $\operatorname{Spec} B$

where B is a flat D-algebra with $B \otimes_D k \simeq B_0$

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & \\ & & I' \arrow[d] & I \arrow[d] & \\ 0 \arrow[r] & R \arrow[r] \arrow[d] & R[t] \arrow[r] \arrow[d, shift right, "f"'] \arrow[d, shift left, "g"] & R = k[x_1, \dots x_n] \arrow[r] \arrow[d, "\pi"] & 0 \\ 0 \arrow[r] & B_0 \arrow[r, "t"] \arrow[d] & B \arrow[r] \arrow[d] & B_0 \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

look for B fitting in the diagram

$$0 \longrightarrow k \xrightarrow{t} D \longrightarrow k \to 0$$

Ambiguity: we can lift π by different maps f, g

$$f, g : R \longrightarrow B_0$$

Lemma: $f - g \in \operatorname{Der}(R; B_0) = \operatorname{Hom}(\Omega_{R/k}, B_0)$

get a map:

$$\to T^0_{B/k}(B_0) \to \operatorname{Hom}(\Omega_{R/k}, B_0) \xrightarrow{\varphi} \operatorname{Hom}(I/I^2, B_0) \to T^1_{B/k}(B_0) \to 0$$

Theorem Deformations of $\operatorname{Spec} B_0/k = T^1_{B/k}$

$T^i_{B/k} : B\text{-mod} \to B\text{-mod} \qquad i = 0,1,2$

Schlessinger functors

Recall • X : smooth var $/k$, $Y_{/k} \subset X$ closed subscheme

then $\exists$ $\mathcal{I}/\mathcal{I}^2 \xrightarrow{d} \Omega^1_{X/k} \otimes \mathcal{O}_Y \to \Omega^1_{Y/k} \to 0$

affine case

$\mathrm{Hom}(d) = \varphi$ above

• if Y is smooth $\Leftrightarrow$ sequence is left exact + $\Omega_{Y/k}$ locally free

Theorem: • if $\operatorname{Spec} B_0$ is smooth then $T^1_{B/k}(B_0) = 0$

$$0 \to I/I^2 \to \Omega_{R/k} \otimes B_0 \to \Omega^1_{B_0/k} \to 0$$

$$\cong$$

$$I/I^2 \oplus \Omega^1_{B_0/k}$$

$\Rightarrow \varphi$ is surjective $\Rightarrow T^1_{B/k} = 0$

• if $T^1_{B/k}(M) = 0 \quad \forall\, M \in B\text{-mod}$

$= \operatorname{coker}\left(\mathrm{Hom}(\Omega_{R/k}, M) \xrightarrow{\varphi} \mathrm{Hom}(I/I^2, M)\right)$

then $\operatorname{Spec} B_0$ smooth.

Global case:

Theorem if $X_{0/k}$ is smooth then the set

$$\left\{ \begin{array}{c} \text{deformations of} \\ X_0 \text{ over } D \end{array} \right\} \xleftrightarrow{1:1} H^1(X_0, T_{X_0})$$

Example: X_0 - smooth proj curve of genus g

$$T_{X_0} = \omega^*_{X_0}, \quad \deg T_{X_0} = 2g-2$$

$$H^1(X_0, T_{X_0}) = H^0(X_0, \omega_{X_0} \otimes T^*_{X_0}) = H^0(X_0, \omega_{X_0}^{\otimes 2})$$

RR: $h^0(T_{X_0}) - h^1(T_{X_0}) = 1 - g + \deg(T_{X_0})$

$$g \geq 2 \Rightarrow h^1(T_{X_0}) = g - 1 + (2g-2) = 3g-3$$