Hyperbolic knots are not generic

(Joint work with Andrei Malyutin)

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Introduction and motivation

Our research project relates to the theory of knots and links and addresses the phenomenon of hyperbolicity. The primary goal of our research is to disprove the long-standing hyperbolicity conjecture for knots and links. The hyperbolicity conjecture states that hyperbolic knots and links are generic. More precisely, it states that the percentage of hyperbolic knots (respectively, of hyperbolic links) amongst all of the prime knots (respectively, all of the prime non-split links) of n or fewer crossings approaches 100 as n approaches infinity. This conjecture dates back to Thurston's famous classification theorem, of 1978, stating that the non-torus non-satellite knot is hyperbolic. (This classification was an important step towards Thurston's geometrization conjecture, presented in 1982 and proved by Grigori Perelman in 2003.) At first glance, Thurston's classification looks paradoxical and highly contra-intuitive. Indeed, our intuition says that the subsets of torus, satellite, and hyperbolic knots and links are scarce while the classification theorem says that these subsets form the whole set. The hyperbolicity conjecture suggests that the solution to this paradox is hidden in the concept of hyperbolicity. This suggestion was based on experimental data: it was discovered that hyperbolic objects, though look scarce, are in fact ubiquitous in some cases. For example, it was shown in [2] that the set of prime knots with at most 16 crossings consists of 1 trivial, 20 satellite, 12 torus, and 1 701 903 (almost two million!) hyperbolic knots. Guided by these striking results (and other results of this kind, some of which we list below) many researchers recalibrate their intuition accordingly, and during several last decades it was strongly believed by most knot/link and 3-manifold theorists that the hyperbolicity conjecture is true and finding its proof is just a matter of time.

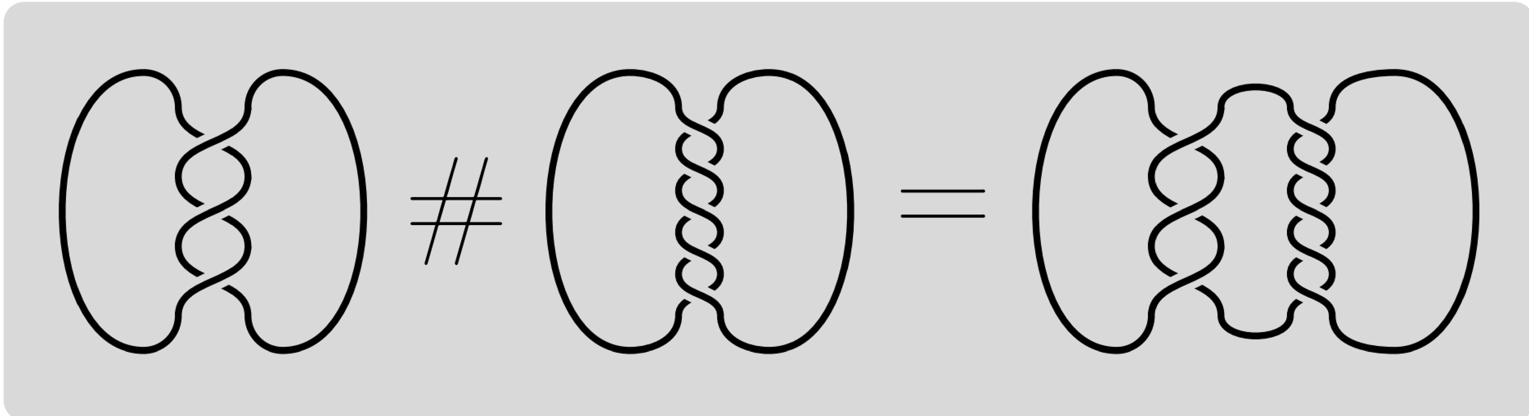
However, recent research results reveal that the situation is more complex. In particular, in a recent work [3] it is shown that the hyperbolicity conjecture contradicts several other plausible conjectures, including the 120-year-old conjecture on additivity of the crossing number of knots under connected sum and the conjecture that the crossing number of a satellite knot is not less than that of its companion. In later work [1] we were able to disprove the hyperbolic conjecture. We have a possible explanation why this result does not contradict the previously obtained demonstrative results on the apparent genericity of hyperbolic knots and links. The answer is that there are no prime satellite knots and links with small crossing number but this fact does not prevent satellites from being typical amongst knots and links of large size. In other words, the statistics of knots is distinct for ones of relatively small and "average" size and for ones of large and huge size.

Basic definitions

Definition 1. The *crossing number* of a knot K (denoted by $\operatorname{cr} K$) is the smallest number of crossings of any diagram of the knot. A diagram of a knot K with precisely $\operatorname{cr} K$ crossings is called *minimal*.

Definition 2. A non-trivial knot is called *prime* if it cannot be written as the connected sum of two non-trivial knots.

An illustration of a connected sum of two prime knots is presented on the figure below.



Connected sum of knots

Definition 3. A non-trivial knot is called *hyperbolic* if its complement $(S^3 \setminus K)$ admits a complete hyperbolic metric of finite volume.

Definition 4. Let $\tilde{K}$ be a knot in a 3-sphere $\tilde{S}^3$ and $\tilde{V}$ an unknotted solid torus in $\tilde{S}^3$ with $\tilde{K} \subset \tilde{V} \subset \tilde{S}^3$. Assume that $\tilde{K}$ in not contained in a 3-ball of $\tilde{V}$. A homeomorphism $h : \tilde{V} \to \hat{V} \subset \hat{S}^3$ onto a tubular neighborhood $\hat{V}$ of a non-trivial knot $\hat{K}$ which maps a meridian of $\tilde{S}^3 \setminus \tilde{V}$ onto a longitude of $\hat{K}$ maps $\tilde{K}$ onto a knot $K = h(\tilde{K})$. The knot K is called a *satellite* of $\hat{K}$.

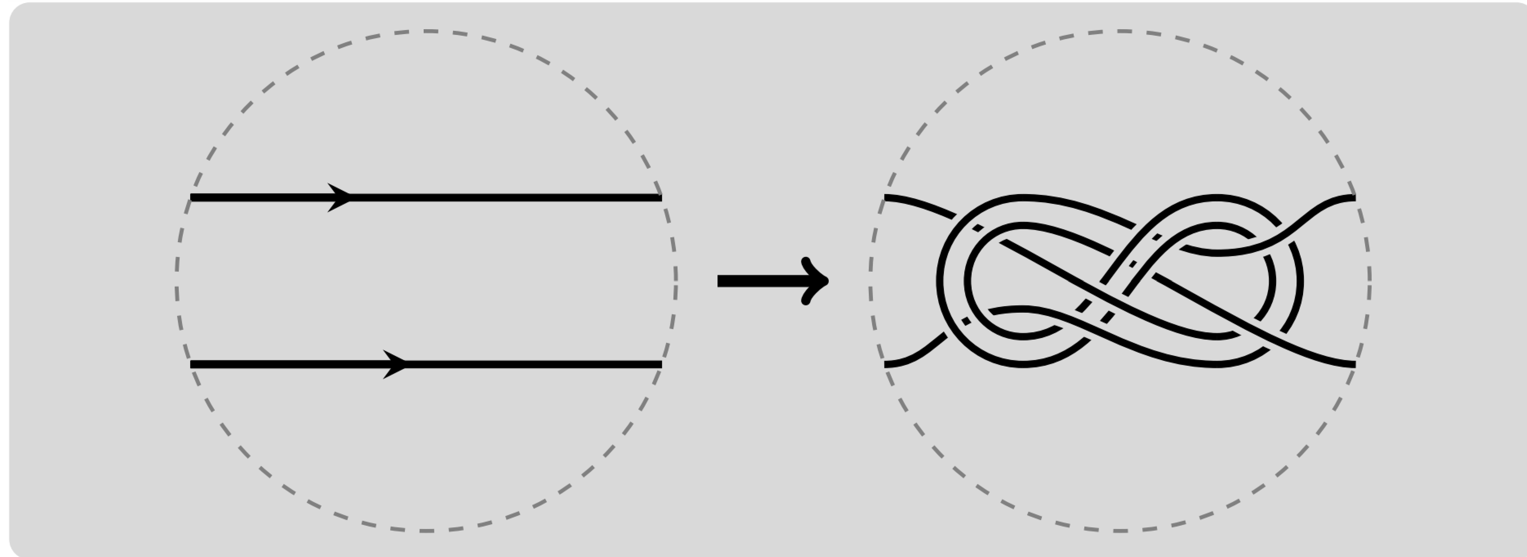
Main result

Theorem 1. *The proportion of hyperbolic knots among all of the prime knots of n or fewer crossings does not converge to 1 as n approaches infinity. Moreover, let P_n (resp., H_n, S_n) denote the number of prime (resp., hyperbolic, prime satellite) knots of n or fewer crossings. Then*

$$\liminf_{n\to\infty} \frac{H_n}{P_n} < 1 - \frac{1}{2 \cdot 10^{17}}.$$

Idea of the proof: We use the fact that the number of prime knot with crossing number at most n grows at most exponential with n.

Let K be a hyperbolic knot with $\operatorname{cr} K = n$, let D be its minimal diagram, and let $\alpha(D)$ be the diagram obtained from D by local move as in figure below.



Double figure-eight move

It is easy to show that a knot $\alpha(K)$ represented by $\alpha(D)$ is a satellite knot. The only things we need to check are:

1. $\alpha(K)$ is prime.
2. If K and K' are two distinct hyperbolic knots then $\alpha(K) \neq \alpha(K')$.

The first item follows from well-known theorems about JSJ-decomposition of knots. The second item uses some technique related to the theory of tangled and some funny geometric ideas. □

Forthcoming Research

In fact we think that a randomly chosen knot with a huge crossing number will almost surely be a satellite knot (with difficult satellite structure related to many knots). We will formulate this as a conjecture.

Conjecture. *The proportion of satellite knots among all of the prime knots of n or fewer crossings converges to 1 as n approaches infinity.*

References

[1] Y. Belousov and A. Malyutin. Hyperbolic knots are not generic, 2019.

[2] J. Hoste, M. Thistlethwaite, and J. Weeks. The first 1,701,936 knots. *The Mathematical Intelligencer*, 20(4):33–48, 1998.

[3] Andrei Malyutin. On the Question of Genericity of Hyperbolic Knots. *International Mathematics Research Notices*, 2020(21):7792–7828, 2018.