SPECIAL MANIFOLDS AND BIRATIONAL CLASSIFICATION, WITH A VIEW TOWARDS ARITHMETICS AND HYPERBOLICITY. YAROSLAV'L SUMMER SCHOOL 2018

FREDERIC CAMPANA

Contents

1. Iitaka dimension of a line bundle

Details and numerous examples in [U]. Let X be a complex and connected projective manifold of dimension n.

Let $\kappa(X,L) := \overline{lim}_{m\to+\infty}\frac{Log(h^0(X,mL)}{Log(m)} \in \{-\infty, 0, 1, ..., n\}$.

Thus:

• $\kappa(X,L) = -\infty$ iff $h^0(X,mL) = 0, \forall m > 0$.

This is the case if $L < 0$, or, more generally, if $L = \mathcal{O}_X(-D)$ for some effective divisor D. And also when X is an elliptic curve, with $c_1(L) = 0$, but L not torsion in $Pic(X)$.

• $\kappa(X,L) = 0$ iff $h^0(X,mL) \leq 1, \forall m > 0$, with equality for some $m > 0$, for example if L is torsion in $Pic(X)$.

• $\kappa(X,L) = n$ iff $mL = A + E$, for some $m > 0$, A ample and E effective.

• $\kappa(X,L) = d \in \{1, ..., n\}$. in the following simple, but typical, example:

$X = Y \times Z, dim(Z) = d, dim(Y) = (n-d), L = p_Z^*(M), M \in Pic(Z)$, ample, $p_Z : X \to Z$ the projection on the second facteur.

Proposition 1.1. *If $\kappa(X,L) = d \geq 0$, for some integer $m > 0$, the rational map $\Phi_{m.L} : X \dashrightarrow \mathbb{P}((H^0(X,m.L)^*)$ associated to the linear system $|m.L|$ is a fibration (with connected fibres), its image $Z = \Phi_{mL}(X)$ has dimension d, is independent of $m >> 0$ suitably chosen (up to birational equivalence), and its generic fibre X_z has $\kappa(X_z, L_{|X_z}) = 0$.*

In particular, if $d = n$, $\Phi_{mL}(X)$ is birational to X.

Said otherwise: $\kappa(X,L) := max_{m>0}\{dim(\Phi_{m.L}(X))\}$.

2. 'Kodaira' dimension

The main case is when $L = K_X := det(\Omega^1_X)$, the canonical line bundle on X. One writes then: $\kappa(X) := \kappa(X, K_X)$[1].

We first start with the case of curves. Then $\kappa(X)$ tells (almost) everything on X, qualitatively.

2.1. **Curves.** The situation is indeed very simple: $\kappa(X) \in \{-\infty, 0, 1\}$ describes X, its topology, fundamental group, and, as seen later, qualitatively, its hyperbolicity and arithmetic properties too.

κ	g	X	$\pi_1(X)$
$-\infty$	$g = 0$	$\mathbb{P}^1$	$\{1\}$
0	$g = 1$	$\mathbb{C}/\Lambda$	$\widetilde{Ab}$
1	$g \geq 2$	$\mathbb{D}/\Gamma$	Γ

• The main objective here will be to define the analogues of these 3 classes in higher dimensions, and to decompose an arbitrary higher dimensional X into its 'components' of the 3 types by a suitable sequence of canonical and functorial fibrations.

[1]In fact introduced by Shafarevich et al.

2.2. **The Iitaka-Moishezon fibration.** The invariant $\kappa(X)$ enjoys several properties:

• It is birational, and preserved by finite étale covers

• Additive for products. Indeed, if $X = Y\times Z$, one has: $h^0(X, mK_X) = h^0(Y, mK_Y) \times h^0(Z, mK_Z), \forall m$, and so: $\kappa(Y \times Z) = \kappa(Y) + \kappa(Z)$.

In particular: $\forall Y_{n-1}$, one has: $\kappa(\mathbb{P}_1 \times Y_{n-1}) = \kappa(Y) + (-\infty) = -\infty$.

Manifolds with $\kappa = -\infty$ are thus not 'indecomposable', and should be decomposed into a 'part' with $\kappa \geq 0$ and a 'part' concentrating the $\kappa = -\infty$ property. The 'rational quotient' map (also called 'MRC' fibration) $r : X \to R$ will realise this decomposition (see 3.7).

• There are 3 fundamental cases:

1. $\kappa(X) = -\infty$.
2. $\kappa(X) = 0$
3. $\kappa(X) = n$.

And $(n-1)$ 'intermediate' cases:

• $0 < \kappa(X) = d < n$. In these cases, X is 'decomposed' as a 'twisted product' of manifolds with $\kappa = 0$ by a manifold Z of lower dimension d by the following fibration J. Indeed:

When $\kappa(X) \geq 0$, the map $J := \Phi_{mK_X} : X \dashrightarrow Z := \Phi_{m.K_X}(X)$, for $m > 0$ suitably large and divisible is birationally well-defined, and may be assumed to be regular (ie: holomorphic). Its generic fibres X_z are then smooth with $\kappa(X_z) = 0$. This is the 'Moishezon-Iitaka' fibration.

We have $\kappa(X_z) = 0$ because $\kappa(X_z, K_{X|X_z}) = 0$, and $K_{X|X_z} = K_{X_z}$ (by the 'Adjunction formula').

When $\kappa(X) = 0$, Z is a point, J does not give any information on X, but when $\kappa(X) = n$, $Z = X$ and J embeds birationally X in the projective space $\mathbb{P}((H^0(X, m.L)^*)$.

Caution: In general, however, $\kappa(Z) < d := dim(Z) = \kappa(X)$. The fibration J thus does not decompose X in parts with $\kappa(X_z) = 0$ and $\kappa(Z) = dim(Z)$. Moreover, J is not defined when $\kappa(X) = -\infty$.

The sought decomposition thus needs further constructions.

The resulting 'classification' table thus looks like this by now, with X_d a smooth hypersurface of degree $d > 0$ in $\mathbb{P}_{n+1}$; by adjunction: $K_{X_d} = \mathcal{O}_X(n + 2 - d)$:

κ	X	$\pi_1(X)$	X_d
$-\infty$	uniruled ?	?	$d \leq (n+1)$
0	K_X birationally torsion ?	$\widetilde{Ab}$?	$d = (n+2)$
$1 < \kappa < n$	$J : X \to Z_\kappa$, $\kappa(X_z) = 0$	?	
n	??	??	$d \geq (n+3)$

The symbol "?" (resp. "??") means that a conjecture exists (resp. that no general structure scheme is known or even possibly expected).

• When $n = 2$ (and to a large extent when $n = 3$), much more is known.

2.3. **Surfaces ($n = 2$).** The 'Kodaira-Enriques-Shafarevich' classification, is displayed in the table below (up to birational equivalence and finite etale covers).

Here, $q = h^1(X, O_X) = h^0(X, \Omega^1_X) = \frac{1}{2}b_1(X)$; $K3$-surfaces are defined by: $q = 0, K_X \cong O_X$. They form a single deformation family containing the smooth quartics in $\mathbb{P}_3$.

κ	q	X(up to bir., étale $\cong$)	$\pi_1(X)$
$-\infty$	$q \geq 0$	$\mathbb{P}^1 \times C_q$	$\pi_1(C_q)$
0	0	$K3$	$\{1\}$
0	2	$(\mathbb{C}^2/\Lambda)$	Λ
1	≥ 0	Elliptic/curve B	$\mathbb{Z}^t \rtimes \pi_1(B), t = 0, 2$
2	≥ 0	??	??

Remark 2.1. *When $\kappa(X) = 1$, we replace X by a suitable finite étale cover in order to eliminate the multiple fibres of the ellliptic fibration.*

3. Uniruledness and rational connectedness

3.1. **Uniruledness.**

Definition 3.1. *X_n is* **uniruled** *if some $f : \mathbb{P}^1 \times Y_{n-1} \dashrightarrow X_n$ rational, dominating, exists.*

Using $Chow(X)$, this also means that X is covered by rational curves (ie: images of non-constant maps from $\mathbb{P}^1$ to X).

Remark 3.2. *If X is uniruled, $\kappa(X) = -\infty$. (True for $V := \mathbb{P}^1 \times Y_{n-1}$, and $\kappa(V) \geq \kappa(X)$ since $f : V \dashrightarrow X$ is dominating).*

The reverse implication is a central conjecture of algebraic geometry (proved for $n \leq 3$).

Conjecture 3.3. *('uniruledness conjecture') If $\kappa(X) = -\infty$, X is uniruled. (Equivalently: $\kappa(X) \geq 0$ if X is not uniruled).*

The following 'approximate' solution is however known (adding a sufficiently ample A to $m.K_X$).

Theorem 3.4. *TFAE:*
1. *X is uniruled.*
2. *$h^0(X, m.K_X + A) = 0$ for $m > m(A)$.*

3.2. **Rational connectedness.**

Definition 3.5. *X_n is said to be rationally connected (RC for short) if, for any $(x, y) \in X \times X$ generic, there is a rational curve containing x and y.*

Example 3.6. *1. $\mathbb{P}_n$ is RC. More generally:*
2. rational $\Longrightarrow$ unirational $\Longrightarrow$ RC.
3. RC surfaces are rational. RC threefolds may not be rational.
4. Fano manifolds (those with $-K_X$ ample) are RC. Thus:
5. $X_d \subset \mathbb{P}_{n+1}$ with $d \leq n+1$ are RC.
6. General $X_4 \subset \mathbb{P}_4$ are expected not to be unirational.
6. RC manifolds are simply-connected.

Uniruled manifolds are canonically decomposed as follows:

Theorem 3.7. *For any X, there exists a (unique) fibration $r : X \to R$ such that:*
1. It fibres are RC
2. R is not uniruled.
The map r is called the 'rational quotient' (or alternatively the MRC fibration) of X.
Conjecture 3.3 implies that:
2^+. $\kappa(R) \geq 0$

The two extreme cases are: X is not uniruled (and $R = X$), and: X is RC (and R is one point).

Theorem 3.8. *TFAE (for A sufficiently ample on X):*
1. X is RC.
2. $h^0(X, \otimes^m(\Omega^1_X) \otimes A) = 0, \forall m \geq m(A)$
3. For any dominating $f : X \dashrightarrow Z$, Z is uniruled.
Conjecture 3.3 implies that this is also equivalent with:
3^+. $\kappa_+(X) = -\infty$ *(see Definition 3.9 below).*

Definition 3.9. $\kappa_+(X) := max\{\kappa(Z), f : X \dashrightarrow Z \text{ dominating}\}$.

The implications $1 \Longrightarrow 2 \Longrightarrow 3$ are easy. Theorem 3.8 gives $3 \Longrightarrow 1$.

4. A failed decomposition attempt

Assume conjecture 3.3. For any X_n, we then have then two maps:
1. $r : X \to R := R_1$ with $\kappa(R) \geq 0$. Thus:
2. $J : R \to J(R) := X_{(1)}$ is well-defined.
The map $J \circ r : X \to X_{(1)}$ is thus well-defined for any X. Moreover:
• $X = X_{(1)}$ if and only if X is of general type.
• Otherwise, $n_1 := dim(X_{(1)}) < dim(X) = n := n_0$.
We can thus iterate the fibration $J \circ r$, applying it successively to $X, X_{(1)}, ..., X_{(k)}$ as long as $n_0 > n_1 > ... > n_k = n_{k+1}$
We have $k \leq n$, since $0 \leq n_k \leq n - k$.

In other words: $(J \circ r)^n : X \to X_{(n)}$ is a fibration such that:
1. $X_{(n)}$ is of general type (possibly a point)
2. The fibres of $(J \circ r)^n$ are towers of fibrations with fibres having either $\kappa^+ = -\infty$ or $\kappa = 0$.
Its fibres are in fact 'very weakly special' in the following sense:

Definition 4.1. *X is 'very weakly special' (vwS for short) if there is no fibration $f : X \dashrightarrow Z_p, p > 0$ with Z of general type.*

Proposition 4.2. Assume Conjecture 3.3. *The fibres of $c_w := (j \circ r)^n : X \to X_{(n)}$ are very weakly special. Its base $X_{(n)}$ is of general type. This is the unique map with these two properties.*

We call the map c_w the 'weak core map' of X.

The proof rests on the following 3 properties:
1. X is vwS if $\kappa_+(X) = -\infty$ (by definition)
2. X is vwS if $\kappa(X) = 0$ (uses Corollary 4.4)
3. If $f : X \to Z$ and $X \to Y$ are fibrations with Z of general type and X_y vwS, then f factorizes through g (ie: $\exists h : Y \to Z$ such that $f = (h \circ g)$).

This last property implies the uniqueness of c_w.

4.1. **Inverting the decomposition.** More importantly, the 'weak core map' can be obtained directly, without using neither Conjecture 3.3, nor the fibrations r and j, but only the following Theorem 4.3 (see Corollary 4.5).

Theorem 4.3. *([V]) Let $f : X \to Z$ be a fibration, with Z of general type (ie: $\kappa(Z) = dim(Z)$). Then $\kappa(X) = \kappa(X_z) + dim(Z)$.*

Corollary 4.4. *Let $f : X \to Z$ be a fibration with $\kappa(X) = 0$. Then $\kappa(Z) < dim(Z)$ if $dim(Z) > 0$.*

Corollary 4.5. *For any X, there is a unique fibration $c_w : X \to C_w$ such that: its fibres are vwS, and C_w is of general type.*

We need Conjecture 3.3 in order to get the decomposition $c_w = (j \circ r)^n$.

4.2. **Etale covers do not preserve the weak core map.** This seemingly satisfactory decomposition result is, however, not preserved by finite étale maps. This is a major failure.

Definition 4.6. *X is 'weakly special' (wS for short) if every finite étale cover of X is very weakly special.*

Example 4.7. *We now give a very simple example of a surface which is vwS, but not wS.*

Let $X_0 = C \times E$, with C a hyperelliptic curve of genus $g \geq 2$, and hyperelliptic involution $h : C \to C$, and quotient $\pi : C \to C / < h > \cong \mathbb{P}^1$, where E is an elliptic curve equipped with translation τ of order 2. Let the fixpoint free diagonal involution $h \times \tau : X_0 \to X_0$ operate on X_0, and let X be the (étale) quotient $p : X_0 \to X := X_0 / < h \times \tau >$.

Thus X is vwS, but not wS.

4.3. **Correcting the failed attempt.** In order to correct the above decomposition $c = (j \circ r)^n$, we shall introduce a stronger notion of specialness:

Definition 4.8. *X is 'special' if* $\kappa(X, L) < p, \forall L \subset \Omega^p_X$ *of rank* 1*, and any* $p > 0$.

Remark 4.9. *If* $f : X \to Z_p, p > 0$ *is a fibration with Z of general type, then* $L_f := f^*(K_Z) \subset \Omega^p_X$ *is a rank* 1 *subsheaf with* $\kappa(X, L_f) = p$. *Thus Special implies very weakly special. We shall see (Corollary 8.2) that Special implies weakly special as well (but the converse does not hold in dimension* 3 *or more (see §.13)).*

The important feature of this new definition is that even if the base of $f : X \to Z$ *is not of general type, the* **saturation** L_f *of* $f^*(K_Z)$ *in* Ω^p_X *may have* $\kappa(X, L_f) = p$. *This is precisely what happens in the example 4.7 above. Moreover,* $\kappa(X, L_f) = \kappa(Z, K_Z + D_f)$*, where the pair* (Z, D_f) *is the 'orbifold base' of f, encoding the multiple fibres of f (in Example 4.7, these are the* $2g+2$ *points of* $\mathbb{P}_1$ *over which* π *ramifies (to order* 2*)).*

The decomposition we are aiming at will be achieved by using the notion of specialness of 4.8 in two steps:

- A first fibration $c : X_n \to C$ (the 'core map') will separate the two opposite 'parts' of X: its fibres (which are 'special'), and its 'orbifold base' (which is of general type). (see §. 8)
- The second step (still depending on conjecture $C^{orb}_{n,m}$) decomposes $c = (j \circ r)^n$ as a sequence of canonical and functorial fibrations r and j. The fibrations r (resp. j) are 'orbifold analogues' of the previous ones, and have 'orbifold' fibres generalising rationally connected (resp. $\kappa = 0$) manifolds respectively.(see §10)

5. Bogomolov sheaves

Theorem 5.1. *([Bog]) Let* $\mathcal{L} \subset \Omega^p_X$ *a rank* 1 *coherent subsheaf. Then:*

1. $\kappa(X, \mathcal{L}) \leq p$.

2. If $\kappa(X, \mathcal{L}) = p$*, there exists a (unique) fibration* $f : X \to Y_p$ *such that* $\mathcal{L} = f^*(K_Y)$ *generically on X. We say that* $\mathcal{L}$ *is a 'Bogomolov sheaf' on X if, moreover,* $p > 0$.

Thus X is 'Special' (as in definition 4.8) means that X does not carry any 'Bogomolov sheaf'.

Remark 5.2. *If* $\mathcal{L}$ *is a Bogomolov, sheaf,* $f^*(K_Y) \subsetneq \mathcal{L}$*, et* $\kappa(Y) < p$*, in general. The difference comes from the 'orbifold base' defined below.*

Example 5.3. *Let X be as in 4.7 above. It is not special. Indeed: its Iitaka fibration* $J : X \to C/ <h>= \mathbb{P}_1$ *has smooth fibres isomorphic to E, with* $2(g + 2)$ *double fibres isomorphic to* $E/ <\tau>$ *over the ramification points* $a_j, j = 1, \ldots, 2g + 2$ *of* $\pi : C \to \mathbb{P}_1$. *Define the* *'orbifold base' of J to be: $(\mathbb{P}_1, D_J)$, with $D_J := \sum_{j=1}^{j=2g+2}(1-\frac{1}{2}).\{a_j\}$, and its canonical bundle to be $L_J := K_{\mathbb{P}_1} + D_J$, so that $\pi^*(L_J) = K_C$. A local computation shows that $J^*(L_J) \subset \Omega^1_X$, and $\kappa(X, J^*(L_J)) = \kappa(\mathbb{P}_1, L_J) = 1$. Thus $J^*(L_J)$ is a Bogomolov sheaf (with $p = 1$), and X is not special.*

This local computation reduces to: $\frac{du}{u^{1/2}} = 2.dx$ if $u = x^2$.

6. Orbifold base of a fibration

6.1. **Orbifold pairs, invariants.** An 'orbifold pair' (X, D) consists of a smooth connected complex projective manifold X_n, together with $D := \sum_{j\in J} c_j.D_j$, a $\mathbb{Q}$-divisor with irreducible components D_j equipped with rational coefficients $c_j \in]0, 1]$.

We also write $c_j := (1-\frac{1}{m_j})$, with $m_j := (1-c_j)^{-1} \in \mathbb{Q}\cap]1, +\infty[\cup+\infty$, called 'the multiplicity' of D_j in D. For any irreducible divisor $F \subset X$, we define $m_D(F) = 1$ if F is not a component of D, and $m_D(F) := m_j$ if $F = D_j$, so that $D = \sum_{F\subset X}(1-\frac{1}{m_F(D)}).F$, also.

We say that (X, D) is 'smooth' if $Supp(D) := \cup D_j$ is SNC, and that it is 'integral' if the $m_j's$ are all in $\mathbb{Z}\cup+\infty$.

• The canonical bundle of (X, D) is defined to be: $K_X + D$, its 'Kodaira' dimension is $\kappa(X, D) := \kappa(X, K_X + D) \geq \kappa(X)$.

6.2. **Orbifold base of a fibration.** Let $f : X \to Y$ be a fibration, with Y smooth. For any irreducible divisor $E \subset Y$, let $f^*(E) := \sum_{k\in K} t_k.F_k + R$, the $F_k's$ being the components of $f^{-1}(E)$ mapped surjectively onto E by f, while R is an f-exceptional divisor of X (ie: $f(R) \subsetneq E$).

Define $m_f(E) := inf\{t_k, k \in K\}$. This is the multiplicity[2] of the generic fibre of f over E. One has: $m_f(E) = 1$, for all but finitely many $E's$. The sum $D_f := \sum_{E\subset Y}(1-\frac{1}{m_f(E)}).E$ is thus finite.

Definition 6.1. *The 'orbifold base' of f is (Y, D_f). Its Kodaira dimension is thus: $\kappa(Y, K_Y + D_f)$.*

Remark 6.2. *More generally, if X is itself equipped with an orbifold divisor D, we shall, for f, Y, E, F_k, t_k as above, define $m_{f,D}(E) := \inf\{t_k.m_D(F_k), k \in K\}$, and the orbifold base $(Y, D_{f,D})$ of $f : (X, D) \to Y$ to be $(Y, D_{f,D})$, with: $D_{f,D} := \sum_{E\subset Y}(1-\frac{1}{m_{f,D}(E)}).E$.*

Example 6.3. *X_y the generic fibre of a fibration $f : X \to Y$, then:*

1. $D_f = 0$ if X_y is RC.

2.If X_y is an abelian variety, then $D_f = D^*_f$, the 'classical' orbifold base of f (see footnote).

[2]Classically, one uses $gcd\{tk\}$, instead of the inf, which leads to the 'classical' orbifold base (Y, D^*_f) of f. One reason to deviate from this choice is given in 6.5.

6.3. **Birational equivalence, neat models of fibrations.** If $f' : X' \to Y'$ is birationally equivalent to f, which is denoted $f' \cong f$, then $\kappa(Y, D_f) \neq \kappa(Y', D_{f'})$ in general.

The birational notion is $\kappa(X, f) := min_{f' \cong f}\{\kappa(Y', K'_Y + D_{f'})\}$, which permits to define $\kappa(X, f)$ even when f is only a rational fibration.

However, $\kappa(Y, D_f) = \kappa(X, f)$ whenever $f : X \to Y$ is 'neat' (ie: obtained by suitable base change and flattening). We assume in the sequel that all our maps are either neat or replaced by a neat birational model. We then speak of the 'neat orbifold base' $(Z, D_Z = D_f)$ of a rational fibration f

6.4. **Orbifold base and saturation of differentials.**

Theorem 6.4. *For any fibration $f : X \to Y_p, p > 0$, if $\mathcal{L}$ is the saturation $f^*(K_Y)^{sat}$ of $f^*(K_Y)$ in Ω^p_X, then: $\kappa(X, \mathcal{L}) = \kappa(Y, K_Y + D_f)$.*

In particular, $\mathcal{L}$ is a Bogomolov sheaf if and only if (Y, D_f) is of general type. Conversely, if $\mathcal{L}$ is a Bogomolov sheaf, it is of the preceding form $f^*(K_Y)^{sat}$ for a unique f.

Remark 6.5. *Theorem 6.4 says that: $\kappa(X, f^*(K_Y)^{sat}) - \kappa(X, f^*(K_Y)) = \kappa(Y, D_f) - \kappa(Y)$. This equality does not hold in general with the 'classical' multiplicities.*

6.5. $C^{orb}_{n,m}$ **Conjecture.**

Conjecture 6.6. *($C^{orb}_{n,m}$) If $f : (X, D) \to Z_p$ is'neat', then:*
$\kappa(X, D) \geq \kappa(X_z, D_z) + \kappa(Z, D_{f,D})$.
In particular: $\kappa(X) \geq \kappa(X_z) + \kappa(Z, D_f) \geq \kappa(X_z) + \kappa(Z)$.

When $D = 0 = D_Z$, this is Iitaka's $C_{n,m}$ conjecture.
A partial solution is given by the:

Theorem 6.7. *If $\kappa(Z, D_Z) = p := dim(Z)$, one has:*
$\kappa(X, D) = \kappa(X_z, D_z) + \kappa(Z, D_Z) = \kappa(X_z, D_z) + p$
Thus: $\kappa(X) = \kappa(X_z) + p$ if $\kappa(Z) = p$.

Theorem 6.7 generalizes Theorem 4.3 to the orbifold context.
This generalisation considerably extends the range of applications of this former result, especially when $\kappa(X_z) = \infty$, or $\kappa(Z) = -\infty$.

Corollary 6.8. *If $\kappa(X, D) = 0$, , one has: $\kappa(Z, D_Z) < p$ for every $f : (X, D) \to Zp, p > 0$, rational dominant, with $D_Z := D_{f,D}$.*

In particular, X is special if $\kappa(X) = 0$.

7. Special Manifolds

Recall:

Definition 7.1. *X_n is 'special (or: 'of special type") if there is no Bogomolov sheaf on X. Equivalently, for any fibration $f : X \dashrightarrow Z_p$, $p > 0$, its 'neat' orbifold base (Z, D_Z) is not of general type.*

Example 7.2. *0. If X is special, it is 'very weakly special'.*

1. A curve X_1 is special if and only if either rational or elliptic.

2. If $f : X \dashrightarrow Y$ is dominating, and if X is special, so is Y.

3. If $X' \to X$ is etale finite and if X est special, then X' too (proof based on Theorem 6.7, surprisingly difficult).

4. If X is RC, it is special.

5. If $\kappa(X) = 0$, X is special (by Corollary 6.8).

Particular (and easier) cases: X is an abelian variety, $c_1(X) = 0$.

6. For any $k, n, -\infty \leq k < n$, there exist special X_n with $\kappa(X_n) = k$.

7. If $f : \mathbb{C}^n \dashrightarrow X_n$ is meromorphic (possibly transcendental) and non-degenerate, X is special ('orbifold version of Kobayashi-Ochiai').

8. A surface X_2 is special if and only if $\kappa(X) < 2$ and $\pi_1(X)$ is almost abelian.

Indeed: when $\kappa = -\infty, 0$, this is clear from classification.

When $\kappa = 1$, after a suitable finite étale cover, the elliptic fibration $J : X \to B$ has no multiple fibre[3], and X is then special if and only if so is B. And thus $\pi_1(X)$ is almost abelian if and only if $g(B) \leq 1$, because of the exact sequence of groups: $\mathbb{Z}^{\oplus 2} \to \pi_1(X) \to \pi_1(B) \to 1$ (if $g(B) = 1$, further arguments are required).

9. There is no such simple characterisation when $n \geq 3$.

Conjecture 7.3. *1. If X is special, $\pi_1(X)$ is almost abelian.*

2. Being special is stable by deformation and specialisation.

This is true if $n = 2$ by 7.2(8), but is unknown if $n > 2$.

8. The 'core map'

Theorem 8.1. *For any X, there exists a unique fibration $c = c_X : X \to C = C_X$ such that:*

1. Its 'general' fibres are special.

2. Its orbifold base (C, D_c) is of general type.

This (almost holomorphic) fibration is the 'core map' (of X).

This map is functorial: if $f : X \dashrightarrow Y$ is dominant, there is a unique map $c_f : C_X \to C_Y$ such that $c_Y \circ f = c_f \circ c_X$.

The extreme cases are when X is of general type (and then $X = C$), and when X is special (and then C is a point). In the intermediate cases, c thus 'splits' X into its antithetic 'parts': special (the fibres) and of general type (the orbifold base).

Idea of proof: If X is special, c is the map onto a point. Otherwise, c is the fibration associated to a Bogomolov sheaf $\mathcal{L} \subset \Omega^p_X$ with maximal $p > 0$. A suitable application of Theorem 6.7 and induction on $n = dim(X)$ show that its general fibres are special, and that this $\mathcal{L}$ is unique □

[3]Except in the simple case where $B = \mathbb{P}_1$ and D_f is supported on two points or less.

Corollary 8.2. *If $u : X' \to X$ is finite étale, $c_u : C' \to C$ is finite (ramified, but étale in an orbifold sense).*

In particular, X' is special if so is X.

Corollary 8.3. *X is special if and only if any two of its generic points can be joined by a chain of special subvarieties (ie: varieties with special desingularisations).*

Remark 8.4. *1. A singular variety need not be special if any two of its points are joined by a chain of special subvarieties (cones over a manifold of general type)*

2. A special X may contain no strict special subvariety (simple abelian varieties, possibly the 'general' smooth quintic threefold).

9. Conjectures in hyperbolicity and arithmetics

Conjecture 9.1. *X is special if and only if :*

(H) equivalently: there is a non-constant holomorphic curve $h : \mathbb{C} \to X$ with Zariski-dense image, or: $\forall x, y \in X$, there exists $h : \mathbb{C} \to X$ with image containing x and y.

(A) (assuming X defined over a number field k): there exists a number field $k' \supset k$ such that $X(k')$ is Zariski dense in X.[4]

These conjectures are motivated by the decomposition $c = (j \circ r)^n$ (see §10), which permits (to a large extent) to reduce them to their orbifold versions about $(X, D)'s$ which are either of general type, or with $\kappa = 0$, or $\kappa_+ = -\infty$). We shall explain this in §. 11.

Using the core map $c : X \to C$, and an orbifold version of Lang's conjectures, we get (see §. 11 for some details):

Conjecture 9.2. *There is an algebraic subset $W \subsetneq C$ such that:*

(H') Any non-constant holomorphic map $h : \mathbb{C} \to X$ has image contained in $c^{-1}(W)$.

(A') (assuming X defined over a number field k): for any number field $k' \supset k$, $[c(X(k')) \cap (C - W)]$ is finite.

Recall that Lang's conjectures asserts these when X is of general type (so that $c = id_X$, then).

10. The decomposition $c = (J \circ r)^n$ of the 'core map'

Let (X, D) be a smooth orbifold pair, let $K_X + D$ be its canonical bundle. If $f : (X, D) \dashrightarrow Z$ is a fibration, we denote the orbifold base of any of its 'neat' birational models by $(Z, D_Z := D_{f,D})$, as in Remark 6.2.

• If $\kappa(X, K_X + D) \geq 0$, the Moishezon-Iitaka fibration is defined as when $D = 0$, using the linear system $m.(K_X + D)$ for $m > 0$

[4]This property is called 'potential density' by geometric arithmeticians.

large and divisible. It gives surjective fibration $J : (X, D) \to Z$ with $dim(Z) = \kappa(X, K_X + D)$ and $\kappa(X_z, K_{X_z} + D_{|X_z}) = 0$ for its generic orbifold fibre.

• The extension of the 'rational quotient' r to smooth orbifold pairs (X, D) is not so immediate. We shall use the simplest version, directly generalising the case $D = 0$.

Definition 10.1. *$\kappa_+(X, D) = -\infty$ if $\kappa(Z, D_Z) = -\infty$ for any rational (neat model of any) dominant fibration $f : (X, D) \dashrightarrow Z_p, p > 0$.*

Remark 10.2. *Conjecturally, the condition $\kappa(X, D) = -\infty$ (resp. $\kappa_+(X, D) = -\infty$) should be equivalent to the fact that (X, D) is unirruled (resp. rationally connected), the definitions being the same as when $D = 0$, but replacing rational curves by D-rational curves defined in 11.3 below. Hence the terminology.*

Proposition 10.3. *Assume $C^{orb}_{n,m}$. For any smooth (X, D), there is a unique[5] fibration $r : (X, D) \to R$ (its 'κ-rational quotient'), such that:*
1. *its general orbifold fibres (X_r, D_r) have $\kappa_+ = -\infty$.*
2. *its (neat) orbifold base (R, D_R) has: $\kappa(R, D_R) \geq 0$.*

We thus get, in complete analogy with the 'very weakly special' case:

Theorem 10.4. *Assume $C^{orb}_{n,m}$ (used to define r).We then have, for any smooth (X, D):*
1. *The composition $(J \circ r) : (X, D) \to (X_1, D_1)$ is well defined.*
2. *So is, for every $k \geq 0$, its k-th iterate $(J \circ r)^k : (X, D) \to (X_k, D_k)$.*
3. *The orbifold fibres of $(J \circ r)^k$ are special, $\forall k \geq 0$.*
4. *$c = (J \circ r)^n$, $n := dim(X)$*

Corollary 10.5. *X is special if and only if it is a tower of fibrations with orbifold fibres having either $\kappa_+ = -\infty$, or $\kappa = 0$.*

Remark 10.6. *This permits to essentially 'reduce' conjectures (such as 7.3) to the same conjectures, but for smooth orbifold pairs having either $\kappa_+ = -\infty$, or $\kappa = 0$.*

Example 10.7. *In order to show that $\pi_1(X)$ is almost abelian if X is special (the 'Abelianity conjecture'), it is sufficient to show that $\pi_1(X, D)$ is almost abelian when $\kappa(X, D) = 0$ and when $\kappa_+(X, D) = -\infty$ for (X, D) smooth and integral. Here, $\pi_1(X, D)$ is the quotient of $\pi_1(X - D)$ by the normal subgroup generated by the m_j^{th} powers of all the small loops γ_j around the $D_j's$.*

The Abelianity conjecture is established when $\pi_1(X)$ has a faithful representation in some $Gl(N, \mathbb{C})$.

[5] Up to birational equivalence.

10.1. **The decomposition $c = (j \circ r)^2$ for surfaces.** We shall give the first step at which the sequence: $id_X, r, j \circ r, r \circ j \circ r, \dots$ stabilises (which is c), and the corresponding decreasing sequence of dimensions of the images, X being special if and only if the last term is 0.

1. $\kappa(X) = -\infty$. Apply $r : X \to R$. There are 3 cases:

1. 1. R is a point: $c = r$, $\{2, 0\}$

1.2. R is a curve and $g(R) = 1$: $c = (j \circ r)$, $\{2, 1, 0\}$

1.3. R is a curve and $g(R) \geq 2$. $c = r$, $\{2, 1\}$

2. $\kappa = 0$. $R = X$, X_1 is a point. $c = (j \circ r)$, $\{2, 2, 0\}$

3. $\kappa(X) = 1$. $R = X$, $X_1 = B$ is a curve. Let (B, D_J) be the orbifold base of $J : X \to B$.There are again 3 cases:

3.1. $\kappa(B, D_J) = -\infty$. $c = (r \circ j \circ r)$, $\{2, 2, 1, 0\}$

3.2. $\kappa(B, D_J) = 0$: $c = (j \circ r)^2$, $\{2, 2, 1, 1, 0\}$

3.3. $\kappa(B, D_J) = 1$: $c = (j \circ r)$, $\{2, 2, 1\}$.

4. $\kappa(X) = 2$: $c = id_X$, $\{2\}$.

11. Orbifold morphisms and orbifold hyperbolicity.

Let C be a smooth connected complex analytic curve. The most interesting cases are when C is either $\mathbb{D}$, $\mathbb{C}$, or $\mathbb{P}_1$. Although the notion of orbifold morphism can be defined more generally in the category of orbifolds, we shall define it only for holomorphic maps $h : C \to (X, D)$ from C to a smooth orbifold pair $(X, D = \sum_j (1 - \frac{1}{m_j}).D_j)$.

Definition 11.1. *The holomorphic map $h : C \to (X, D)$ is an orbifold morphism if $h(C) \subsetneq D$, and if, for any j: $h^*(D_j) \geq m_j.h^{-1}(D_j)$.*

This means that each time $h(C)$ meets some D_j, the order of contact has to be at least[6] m_j.

We have the following important functoriality of orbifold morphisms:

Example 11.2. *Let $f : (X, D) \to (Z, D_Z)$ be a neat fibration onto its orbifold base. For any holomorphic $h : C \to X$, $f \circ h : C \to (Z, D_Z)$ is an orbifold morphism (if $f \circ h(C) \subsetneq D_Z$).*

There is a similar statement in arithmetic.

Definition 11.3. *Let (X, D) be smooth. A D-rational curve (resp. a D-entire curve) is an orbifold morphism $h : C \to (X, D)$ with $C = \mathbb{P}_1$ (resp. $C = \mathbb{C}$).*

We say that (X, D) is uniruled (resp. RC) if there exists a D-rational curve through x (resp. (x, y)) for any generic $x \in X$ (resp. $(x, y) \in X \times X$).

Extending conjectures which are (more or less) standard when $D = 0$, we get the following conjectures, which (essentially) imply and motivate the conjectures 9.1 and 9.2:

[6]The notion of 'classical" orbifold morphism is that this order of contact be divisble by m_j.

Conjecture 11.4. *1. If $\kappa(X, D) = -\infty$ (resp. $\kappa_+(X, D) = -\infty$), then (X, D) is uniruled (resp. rationally connected).*

2. If If $\kappa(X, D) = 0$, then $\forall x, y \in X$, there exists a D-entire curve containing x and y.

3. If (X, D) is of general type, there exists $W \subsetneq X$ algebraic such that any D-entire curve is contained in W.

These conjectures have arithmetic (and function field) analogues, according to S. Lang. We give in the next section the simplest statement: for $X = \mathbb{P}_1$, and $k = \mathbb{Q}$.

12. Orbifold Mordell Conjecture

Let $(\mathbb{P}_1, D)$, where $D := (1 - \frac{1}{r}).\{0\} + (1 - \frac{1}{s}).\{1\} + (1 - \frac{1}{t}).\{\infty\}$, with r, s, t positive integers, and assume that $(\frac{1}{r} + \frac{1}{s} + \frac{1}{t}) < 1$ (ie: that $(\mathbb{P}_1, D)$ is of general type).

For any non-zero integer $0 \neq u \in \mathbb{Z}$, its 'radical' $rad(u)$ is the product of the prime numbers which divide u. And u is said to be 'r-full' if $rad(u)^r$ divides u. This means that the exponent in the prime decomposition of u of any prime p dividing u is at least r.

For u to be an exact r-th power[7], these exponents have to be all divisible by r. So r-th powers are r-full, but not conversely.

Definition 12.1. *A rational number $x = \frac{u}{v}$ in lowest terms is D-integral if (and only if) u is r-full, v is t-full, and $u - v$ is s-full.*

In other words, the integral points of $(\mathbb{P}_1, D)$ are the solutions of the equation $u = v + w$, with u, v, w respectively r, t, s-full. These integral points thus generalise the integral solutions of the equation $a^r = b^t + c^s$.

Conjecture 12.2. *('Orbifold Mordell Conjecture' for $(\mathbb{P}_1, D), k = \mathbb{Q}$) The number of D-integral points of $(\mathbb{P}_1, D)$ above is finite.*

Remark 12.3. *This conjecture is implied by the 'abc' conjecture. The finiteness of the set of integral solutions of the equation $a^r = b^t + c^s$ is, by contrast, known (it is reduced to Falting's theorem by a covering trick, by Darmon-Granville).*

Geometric interpretation: The D-integral points are thus nothing else then the orbifold morphisms from $Spec(\mathbb{Z})$ to the arithmetic orbifold surface $(\mathbb{P}_1, D)(\mathbb{Z})$. Indeed:

A rational number $x = \frac{u}{v}$ in lowest terms as before is seen as a section of the projection of the arithmetic surface $\mathbb{P}_1(\mathbb{Z})$ sitting over $Spec(\mathbb{Z})$ with fibre over each prime p equal to the reduction $\mathbb{P}_1(\mathbb{Z}_p)$ of $\mathbb{P}_1(\mathbb{Z})$ modulo p. The point of the section x lying over p is then the reduction x_p of x modulo p.

The orbifold divisor $D(\mathbb{Z})$ then consists of the 3 disjoint sections $0, 1, \infty$ over $Spec(\mathbb{Z})$. The order of contact of the section x with the

[7] This corresponds to the notion of 'classical' integral point.

section 1 say at x_p is then equal to the exponent of p in the decomposition of $(x-1)=\frac{u-v}{v}$, seen as the vanishing order of the function $(x-1)$ at p. The interpretation is similar for the intersections of x with the sections $0,\infty$.

Remark 12.4. *When $(\frac{1}{r}+\frac{1}{s}+\frac{1}{t})\geq 1$, Conjecture 9.1(A) claims that the integral points of this orbifold should be infinite. This is true even for the 'classical' integral points, by using suitable coverings (by either elliptic or rational curves). The only hard case is when $(r,s,t)=(2,3,5)$, where the icosahedral cover of Felix Klein is required.*

There is a (complex) function field version of the preceding conjecture, which is known.

13. Specialness vs weak specialness

Recall that a smooth projective X is said to be weakly special if no finite étale cover $X'\to X$ admits a rational dominant $f':X'\dashrightarrow Z'_p, p>0$ to some Z' of general type.

We saw that specialness implies weak specialness, and both notions agree in dimension at most 2, after 7.2(8).

• We shall now give examples of weakly special but non special threefolds, extending a construction initially due to Bogomolov-Tschinkel.

Theorem 13.1. *There exists a simply-connected smooth projective threefold X together with an equidimensional elliptic fibration $F:X\to S$ on a smooth surface S such that:*

1. X is simply-connected.

2. $\kappa(S)\leq 1$ (and any of the three values $-\infty,0,1$ can be chosen)

3. $\kappa(S,D_F)=2$ if (S,D_F) is the orbifold base of F (which is neat and smooth).

Any such $F:X\to S$ with these properties is such that X is weakly special, but not special.

Proof. Let us prove the last claim: (3) implies that X is not special, and that F is the core map of X. In order to show that X is weakly special, it is sufficient to see that there is not fibration $g:X\dashrightarrow Z$ with Z of general type, and $p:=dim(Z)>0$. Indeed since g has to factorise through F, if $p=2$, $Z=S$, if $p=1$, Z is simply connected hence $\mathbb{P}_1$. Contradiction since both are not of general type.

The recipe for the construction of X needs two 'ingredients':

1. A projective elliptic surface $f:T\to\mathbb{P}_1$ with one fibre $T_1:=f^{-1}(1)$ which is simply-connected, and a multiple smooth fibre $T_0=f^{-1}(0)$ of multiplicity $m>1$. One can obtain such a surface from a logarithmic transform of a projective elliptic surface $T'\to\mathbb{P}_1$ with simply-connected fibre T'_1 and $p_g(T')=0$.

2. A surface $g : S \to \mathbb{P}_1$ with $\kappa(S) \leq 1$ and smooth fibre $S_0 = g^{-1}(0)$ such that $\pi_1(S - S_0) = \{1\}$. This can be constructed from any simply-connected surface S' with $\kappa(S') \leq 1$, by choosing on S' a base-point free ample linear system defined by a smooth ample divisor $D' \subset S'$, and a second generic member D" of this linear system which meets transversally D' at $d := (D')^2$ distinct points, and such that, moreover, $\kappa(S', K'_S + (1 - \frac{1}{m}).D') = 2$.

For example, $S' = \mathbb{P}_2$, and D', D" two generic quartic curves satisfy these conditions.

One then blows-up all points of $D' \cap D$" to obtain S, and $g : S \to \mathbb{P}_1$ is the map defined by the pencil generated by D', D". One takes for $D = S_0$ the strict transform of D' in S. The simple-connectedness of $(S - D)$ is a consequence of a version of Lefschetz theorem.

We now choose $X_3 := S \times_{\mathbb{P}_1} T$, and $F : X \to S$ the first projection.

In order to show that the orbifold base (S, D_F) of $F : X \to S$ is of general type, observe that $F^*(D) = m.F^{-1}(D)$, since $D = g^{-1}(0)$, and $f^{-1}(0) = m.T_0$. Thus $D_F \geq (1 - \frac{1}{m}).D$, and an easy computation shows that $\kappa(S, (1 - \frac{1}{m}).D) = \kappa(S', (1 - \frac{1}{m}).D') = 2$, since $K_S = b^*(K_{S'}) + E$, while $D = b^*(D') - E$, if $b : S \to S'$ is the blow-up and E its reduced exceptional divisor. And so: $K_S + (1 - \frac{1}{m}).D = b^*(K_{S'} + (1 - \frac{1}{m}).D') + \frac{1}{m}.E$. □

Remark 13.2. *For some examples of the above X it is possible to check that the hyperbolicity properties are as stated in Conjecture 9.2(H'). In the arithmetic case, a conjecture of Abramovich-Colliot-Thélène*[8]*, which conflicts with 9.1(A), claims that the above threefolds should be potentially dense when defined over some number field. The present state of arithmetic geometry does not permit to solve the conflict.*

[8] It claims that weak specialness, instead of specialness, implies potential density.